Dynamics of Cylindrical Shells Containing Fluid Flows with a Developing Boundary Layer

M. P. Païdoussis* and A. D. Mateescu†
McGill University, Montreal, Québec, Canada

This paper deals with the dynamics and stability of a thin cylindrical shell, with clamped ends, conveying incompressible viscous fluid. In particular, the case of a developing boundary layer is considered. It is shown that the main unsteady fluid-dynamic terms influencing the dynamics of the system are 1) the unsteady pressure associated with shell oscillations and 2) the unsteady circumferential component of the skin friction stress, coupled with circumferential movements of the shell, expressions for which are obtained for both a laminar and a turbulent boundary layer. The problem is formulated in terms of a unique equation of motion obtained from Flügge's set of three equations, without the intercession of additional simplifications or assumptions, such as those utilized by Donnell and Kempner, for instance. It is shown that the unsteady viscous effects on the dynamics of the problem are generally small and stabilizing. The effect of some system parameters on stability is also investigated.

I. Introduction

THE dynamics of cylindrical shells conveying fluid has been investigated quite extensively,[1-10] ever since it was discovered that, for sufficiently high flow, the system is subject to fluidelastic instabilities.[1] It was found[1,2] that cantilevered shells lose stability by single mode flutter, in both beam- and shell-type modes, while shells with supported ends[2-5] lose stability by divergence, followed by coupled-mode flutter. The theoretical predictions were confirmed by experimental evidence.[2] A comprehensive literature review was undertaken in Ref. 11, where the more complex problem of coaxial shells conveying fluid in the inner shell and within the annulus is treated.

In all these studies the fluid was considered to be inviscid and the flow irrotational, so that the fluid-dynamic forces exerted on the shell could be determined by means of ideal flow theory. The neglect of viscous fluid effects is probably partly due to the fact that, for thick-walled tubular beams (analyzed as Euler-Bernoulli beams) conveying fully developed turbulent flow, surface-traction and pressure-drop effects cancel out in the equations of motion, so that the analysis may be conducted as if the fluid were totally inviscid.[12] However, in the case of thin-walled tubes conveying fluid, where shell-type as well as beam-type modes are involved, the effects of fluid viscosity do generally enter the problem—as will be made clear presently.

The effects of fluid viscosity on the dynamics of the shell may be separated into zero-order and first-order effects. The former arise because of 1) the pressurization at the upstream end or, equivalently, depressurization at the downstream end, associated with the viscous pressure drop in the tube, and 2) the corresponding surface traction. The associated stress resultants acting on the shell are steady (time-independent) and they enter the equations of motion as generalized forces associated with virtual displacements of the shell. These effects have recently been examined[13] and, as expected from physical reasoning (pressurization being by far the dominant effect), were found to be stabilizing—vis-à-vis inviscid flow results—at least for shells with supported ends.

What may be termed as first-order effects are the unsteady viscous effects of real fluids on the dynamics of the system. They have never been considered heretofore and they are the principal consideration of the present paper. This is done by first developing an improved formulation of the problem, from the point of view of computing efficiency and accuracy, involving a unique equation of motion (as opposed to the usual three), and also an improved solution of the unsteady potential flow part of the problem.

Specifically, the problem that will be examined in this paper involves a thin circular shell, clamped at both ends, conveying fluid. Unsteady viscous effects are examined in the special case of a developing boundary layer, either laminar or turbulent.

II. Formulation of the Problem

System Definition and Assumptions

The system consists of a thin cylindrical shell of length L, wall thickness h and mean radius a (Fig. 1). The shell, both ends of which are clamped, ingests fluid at the inlet, at which point $x=0$.

The fluid flow consists of an incompressible core flow, which may be treated as if it were inviscid, and a thin developing boundary layer, which will be considered to be either laminar or turbulent. Only unsteady viscous effects will be considered; the effect of mean steady loads generated by pressure drop in the mean flow, such as were treated

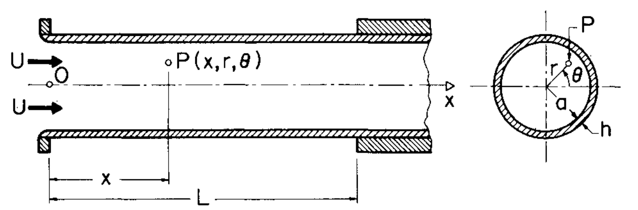

Fig. 1 Geometry of the cylindrical shell conveying axial flow, showing a representative point in the fluid, *P*.

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*Thomas Workman Professor, Department of Mechanical Engineering.

†Formerly Graduate Student, Department of Mechanical Engineering; currently Aerodynamicist at Pratt & Whitney Canada Ltd., Longueuil, Québec, Canada.

previously in Ref. 13, will be ignored in the present study, partly for simplicity and partly in order to highlight those effects that have never been studied heretofore.

The Equations of Motion

Flügge's shell equations of motion[14] will be used in this paper. In dimensionless form, they may be expressed as

$$\left\{\left(\frac{\partial^2}{\partial\xi^2}\right)+\tfrac{1}{2}(1-\nu)(1+k)\left(\frac{\partial^2}{\partial\theta^2}\right)-\left(\frac{\partial^2}{\partial\tau^2}\right)\right\}\bar{u}$$
$$+\left\{\tfrac{1}{2}(1+\nu)\left(\frac{\partial^2}{\partial\xi\partial\theta}\right)\right\}\bar{v}+k\left\{\tfrac{1}{2}(1-\nu)\left(\frac{\partial^3}{\partial\xi\partial\theta^2}\right)\right.$$
$$\left.-\left(\frac{\partial^3}{\partial\xi^3}\right)\right\}\bar{w}=-\left\{\frac{(1-\nu^2)}{E\bar{h}}\right\}\tau_x$$
$$\left\{\tfrac{1}{2}(1+\nu)\left(\frac{\partial^2}{\partial\xi\partial\theta}\right)\right\}\bar{u}+\left\{\tfrac{1}{2}(1-\nu)(1+3k)\left(\frac{\partial^2}{\partial\xi^2}\right)+\left(\frac{\partial^2}{\partial\theta^2}\right)\right.$$
$$\left.-\left(\frac{\partial^2}{\partial\tau^2}\right)\right\}\bar{v}-\left\{\tfrac{1}{2}k(3-\nu)\left(\frac{\partial^3}{\partial\xi^2\partial\theta}\right)\right\}\bar{w}=-\left\{\frac{(1-\nu^2)}{E\bar{h}}\right\}\tau_\theta$$
$$\left\{\nu\left(\frac{\partial}{\partial\xi}\right)+k\left[\tfrac{1}{2}(1-\nu)\left(\frac{\partial^3}{\partial\xi\partial\theta^2}\right)-\left(\frac{\partial^3}{\partial\xi^3}\right)\right]\right\}\bar{u}+\left\{\left(\frac{\partial}{\partial\theta}\right)\right.$$
$$\left.-\tfrac{1}{2}k(3-\nu)\left(\frac{\partial^3}{\partial\xi^2\partial\theta}\right)\right\}\bar{v}+\left\{(1+k)+k\nabla^4+2k\left(\frac{\partial^2}{\partial\theta^2}\right)\right\}\bar{w}$$
$$+\left\{\left(\frac{\partial^2}{\partial\tau^2}\right)\right\}\bar{w}=\left\{\frac{(1-\nu^2)}{E\bar{h}}\right\}(p-p_e) \qquad (1)$$

where $\bar{u}$, $\bar{v}$, $\bar{w}$ are the axial, circumferential, and radial displacements of the mean surface of the shell, nondimensionalized with respect to the shell radius a; ν is Poisson's ratio, and E is Young's modulus of the shell material; $\tau_x(x,\theta,t)$ and $\tau_\theta(x,\theta,t)$ are the axial and circumferential stress resultants, acting tangentially on a unit area of the shell; $p(x,\theta,t)$ and $p_e(x,\theta,t)$ are the internal and external pressures, acting radially on the shell; the dimensionless axial coordinate, ξ, and time, τ, are defined by

$$\xi=x/a, \qquad \tau=\{E/[\rho_s a^2(1-\nu^2)]\}^{1/2}t \qquad (2)$$

where ρ_s is the shell density and $k=(1/12)\bar{h}^2$, with $\bar{h}=h/a$.

This is the usual form of the equations of motion. In this paper this set of three equations is reduced to a single, higher-order equation, involving only the displacement $\bar{w}$. The derivation, involving many stages of successive differentiation and elimination, is unfortunately too long to present here; the interested reader will find a detailed derivation in Ref. 15. The final equation is given in symbolic form below

$$\mathcal{L}[\bar{w},\ \tau_x,\ \tau_\theta,\ p-p_e]=0 \qquad (3)$$

and in full in the Appendix. $\mathcal{L}$ is an eighth-order linear differential operator with respect to ξ, θ, and τ. It should be noted that this equation was obtained, with the intervention of no additional approximations, beyond those already present in Eqs. (1). Thus, the kind of shallow-shell assumption that was introduced by Donnell, for instance, to obtain a much simpler equation, or by Kempner, who neglected the axial and circumferential inertia forces, is entirely absent here.

Solution of the equations of motion will be achieved once τ_x, τ_θ and $p-p_e$ are expressed as functions of $\bar{w}$—to be done later—so that Eq. (3) becomes $\mathcal{L}'[\bar{w}]=0$. Assuming a certain mode of vibration of the shell, defined by the circumferential mode number n, and considering the axial modal shape to be represented by a set of comparison functions, $\phi_m(\xi)$, one may express $\bar{w}$ as

$$\bar{w}(\xi,\theta,\tau)=\sum_m c_m\phi_m(\xi)\cos n\theta\, e^{\lambda\tau} \qquad (4)$$

where λ is a dimensionless eigenvalue of the problem; for $\phi_m(\xi)$ are used the eigenfunctions of a clamped-clamped beam, satisfying the boundary conditions of the shell, and which may be written for convenience, as

$$\phi_m(\xi)=\phi_{mT}(\xi)+\phi_{mH}(\xi)$$
$$\phi_{mT}(\xi)=-\cos(\beta_m\xi/\ell)+\sigma_m\sin(\beta_m\xi/\ell)$$
$$\phi_{mH}(\xi)=\cosh(\beta_m\xi/\ell)-\sigma_m\sinh(\beta_m\xi/\ell) \qquad (5)$$

where β_m are the dimensionless eigenvalues of a clamped-clamped beam and $\ell=L/a$. Then application of the Galerkin method requires that

$$\int_0^1 \mathcal{L}[\bar{w},\ \tau_x,\ \tau_\theta,\ p-p_e]\phi_q(\xi)\mathrm{d}\xi=0, \qquad q=1,2,\ldots \qquad (6)$$

leading to a determinantal equation, nontrivial solution of which yields the eigenvalues of the system.

The fluid dynamic terms will next be determined. The case of wholly inviscid flow will be treated first, and $p-p_e$ obtained—τ_x and τ_θ in this case being zero. The case of viscous flow will be treated in Sec. III.

The Inviscid Potential Flow

Denoting the mean velocity of the incompressible fluid flow by U and the disturbance velocity, associated with shell motions, by $\boldsymbol{q}$, the total flow velocity may be written as

$$V=U\boldsymbol{i}+\boldsymbol{q} \qquad (7)$$

where $\boldsymbol{i}$ is the unit vector in the x direction. Then the Euler equation for the fluid is given by

$$\left(\frac{\partial \boldsymbol{q}}{\partial t}\right)+(U\boldsymbol{i}+\boldsymbol{q})\cdot\nabla\boldsymbol{q}=-(1/\rho)\nabla p \qquad (8)$$

For irrotational flow, $\boldsymbol{q}=\nabla\Phi$ may be introduced, where Φ is the disturbance velocity potential, and the pressure obtained from the Bernoulli equation

$$\left(\frac{\partial\Phi}{\partial t}\right)+\tfrac{1}{2}V^2+p/\rho=\tfrac{1}{2}U^2+p_\infty/\rho \qquad (9)$$

where p_∞ is the static pressure of the undisturbed uniform flow; introducing the small disturbance assumption, this is further reduced to

$$p-p_\infty=-\rho\left[\left(\frac{\partial\Phi}{\partial t}\right)+U\left(\frac{\partial\Phi}{\partial x}\right)\right] \qquad (10)$$

The disturbance velocity, and hence Φ, must satisfy the boundary conditions:

$$\left(\frac{\partial\Phi}{\partial r}\right)\bigg|_{r=a}=\left(\frac{\partial w}{\partial t}\right)+U\left(\frac{\partial w}{\partial x}\right) \qquad (11)$$

on the surface of the shell, where w is the radial displacement of the shell; and, considering a uniform inlet flow

$$\left(\frac{\partial\Phi}{\partial x}\right)\bigg|_{x\to-\infty}=0 \qquad (12)$$

In view of Eq. (11), the form of the solution will be taken to be similar to that of Eq. (4), so that in dimensionless terms

$$\Phi(\xi,\bar{r},\theta,\tau) = aU_{\mathrm{ref}}\phi(\xi,\bar{r})\cos n\theta\, e^{\lambda\tau} \tag{13}$$

where $\phi(\xi,\bar{r})$ is the so-called reduced potential, and

$$\bar{r} = r/a, \qquad U_{\mathrm{ref}} = \{E/[\rho_s(1-\nu^2)]\}^{1/2} \tag{14}$$

Pursuing the solution by standard means, the reduced potential is eventually determined to be

$$\phi(\xi,\bar{r}) = \sum_m c_m\{\eta_m(\bar{r})[\lambda\phi_{mH}(\xi) + \bar{U}\phi'_{mH}(\xi)] + \zeta_m(\bar{r})[\lambda\phi_{mT}(\xi) + \bar{U}\phi'_{mT}(\xi)]\} \tag{15}$$

where

$$\eta_m(\bar{r}) = J_n(\beta_m\bar{r}/\ell)/[nJ_n(\beta_m/\ell) - (\beta_m/\ell)J_{n+1}(\beta_m/\ell)]$$

$$\zeta_m(\bar{r}) = I_n(\beta_m\bar{r}/\ell)/[nI_n(\beta_m/\ell) + (\beta_m/\ell)I_{n+1}(\beta_m/\ell)]$$

$$\bar{U} = U/U_{\mathrm{ref}}, \qquad \ell = L/a \tag{16}$$

where J_n and I_n are the ordinary and modified Bessel functions of the first kind, respectively, and ϕ_{mH} and ϕ_{mT} are defined by Eq. (5).

Utilizing Eq. (10), the unsteady pressure on the shell is found to be

$$p - p_\infty = -\rho U_{\mathrm{ref}}\sum_m c_m\{\lambda^2[\eta_m\phi_{mH}(\xi) + \zeta_m\phi_{mT}(\xi)] + 2\bar{U}\lambda[\eta_m\phi'_{mH}(\xi) + \zeta_m\phi'_{mT}(\xi)] + (\beta_m/\ell)^2\bar{U}^2[\eta_m\phi_{mH}(\xi) + \zeta_m\phi_{mT}(\xi)]\}\cos n\theta\, e^{\lambda\tau} \tag{17}$$

where η_m and ζ_m are the values of $\eta_m(\bar{r})$ and $\zeta_m(\bar{r})$ at $\bar{r}=1$.

The effect of the stationary outer fluid on the dynamics of the shell will be neglected here, for simplicity.

III. The Unsteady Viscous Flow

The unsteady viscous flow in the oscillating shell is governed by the Navier-Stokes equations. The problem at hand, involving a developing boundary layer on the inner surface of the shell, may be reduced, via Prandtl's concept of a thin boundary layer with respect to the shell radius, to the following two problems: 1) the steady viscous flow in the boundary layer, and 2) the unsteady flow in the central core.

The central core flow is quasi-inviscid, because of the relatively small velocity gradients therein; the viscous effects manifest themselves in the boundary conditions, which are applied at the edge of the displacement thickness of the boundary layer, $r = a - \delta^*(x)$, as will be done later in determining $p - p_e$.

The unsteady viscous flow in the boundary layer, wherein velocity gradients are large in the vicinity of the wall, are discussed next. Throughout, in dealing with the unsteady viscous flow, the assumption of small-amplitude oscillations of the shell is made; hence, the unsteady perturbation velocities induced thereby are small, as compared to the undisturbed axial flow velocity U.

The Unsteady Viscous Flow in the Boundary Layer

Considering that the radial and tangential components of the perturbation velocity are small, as compared to the axial component of the flow velocity, V_x, and utilizing the usual assumptions of the boundary-layer concept, the Navier-Stokes equations of the unsteady viscous flow in a *laminar boundary layer* on the shell wall may be written in the linearized form

$$\frac{\partial V_x}{\partial t} + U\frac{\partial V_x}{\partial x} = \frac{\partial V_{x\ell}}{\partial t} + U\frac{\partial V_{x\ell}}{\partial x} + \nu_f\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_x}{\partial r}\right)$$

$$0 \simeq -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

$$\frac{\partial V_\theta}{\partial t} + U\frac{\partial V_\theta}{\partial x} = \frac{\partial V_{\theta\ell}}{\partial t} + U\frac{\partial V_{\theta\ell}}{\partial x} + \nu_f\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_\theta}{\partial r}\right) \tag{18}$$

where $V_{x\ell}$ and $V_{\theta\ell}$ are the values of V_x and V_θ at the interface between the boundary layer and the core flow, and will be varying in time due to shell oscillations; and ν_f is the kinematic viscosity of the fluid. The radial component of the flow velocity at the boundary layer limit is sensibly equal to the radial velocity of the shell; hence, the relative velocity with respect to the shell wall, which determines the extent of viscous effects, does not have a radial component.

Assuming that the displacement thickness $\delta(x)$ of the developing boundary layer is small with respect to the shell radius a, the steady-flow velocity U at the boundary-layer edge can be derived from the continuity equation:

$$U_\ell(x) = U\{a^2/[a - \delta^*(x)]^2\} \simeq U(1 + 2\delta^*/a) \tag{19}$$

The displacement thickness, $\delta^*(x)$, may be derived in various ways.[16,17] Here, because of the complexity of the overall problem at hand, it is desirable to have a simple analytical expression for $\delta^*(x)$; such an expression was derived in Ref. 15, namely

$$\delta^*(x)/a = \tfrac{1}{2}\{1 + [4x/Re_a - 1]/[1 + (12x/Re_a)^{1/2}]\} \tag{20}$$

where Re_a denotes the Reynolds number based on the shell radius, a.

In the assumption of small amplitude oscillations, the unsteady perturbation velocities $V_{x\ell} - U_\ell$ and $V_{\theta\ell}$ are negligible with respect to U_ℓ; hence the magnitude of the fluid velocity V_ℓ relative to the shell surface, considered at the edge of the boundary layer, can be assumed to be invariant in the time since $V_\ell = [V_{x\ell}^2 + V_{\theta\ell}^2]^{1/2} \simeq U_\ell$. However, the direction of the flow velocity V_ℓ is varying in time with the angle β, defined as $\tan\beta \simeq V_{\theta\ell}/V_{x\ell}$.

The circumferential velocity $V_{\theta\ell}$ may be obtained approximately from the value of V_θ in the core flow near the wall ($\bar{r}=1$); i.e., from Eqs. (13) and (15)

$$V_{\theta\ell} \simeq V_\theta(\xi,1,\theta,\tau) = -U_{\mathrm{ref}}n\phi(\xi,1)\sin n\theta\, e^{\lambda\tau} \tag{21}$$

Actually, the circumferential velocity at the boundary-layer edge should be calculated accounting also for the displacement thickness of the boundary layer, $\delta^*(x)$; an iterative process could be used in this respect. However, the effects of the second iteration are expected to be small enough in the framework of the present assumptions, so that this iterative procedure is considered an unnecessary refinement in the present analysis.

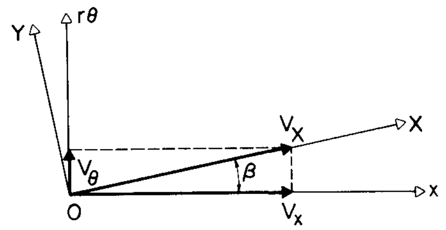

Fig. 2 Diagram showing transformation of coordinates.

At the shell wall, the direction of the flow velocity relative to the x axis is

$$\tan\beta = V_\theta(\xi,1,\theta,\tau)/U_\ell \simeq -[n\phi(\xi,1)/\bar{U}]\sin n\theta\, e^{\lambda\tau} \tag{22}$$

as shown in Fig. 2, where $V_x|_{r=a-\delta^*} \simeq U_\ell$, $U_\ell \simeq U$, and Eq. (16) have been utilized.

Now, introducing the coordinate system (X,Y), as shown in Fig. 2, $V_X = V_x\cos\beta + V_\theta\sin\beta$, which at $r=a-\delta^*(x)$ becomes

$$V_{X\ell} = V_{x\ell}\cos\beta + V_{\theta\ell}\sin\beta \tag{23}$$

hence, Eqs. (18) may now be recast in the new coordinate frame and linearized to give the remarkably simple result

$$0 \simeq \nu_f\left(\frac{1}{r}\right)\left(\frac{\partial}{\partial r}\right)\left[r\left(\frac{\partial V_X}{\partial r}\right)\right] \tag{24}$$

Thus, the problem of unsteady flow in the boundary layer has been reduced to one of quasisteady flow, which may then be treated by standard methods. This was achieved through the original assumption that the flow velocity in the boundary layer is sensibly constant in magnitude, but fluctuating in direction with respect to time, as characterized by the angle β.

Now, proceeding with the method developed in Ref. 15, based on Lighthill's approach to boundary layers on the exterior of bodies of revolution, one obtains

$$\tau_x = \tfrac{1}{2}\rho U^2 C_\ell (1/Re_a x)^{1/2}$$

$$\tau_\theta = \tfrac{1}{2}\rho U^2 C_\ell (1/Re_a x)^{1/2}[-n\phi(\xi,1)/\bar{U}]\sin n\theta\, e^{\lambda\tau} \tag{25}$$

with $C_\ell = 1/\sqrt{3}$. It is noted that τ_x is associated with the steady surface traction on the shell; as in this paper the principal concern is with *unsteady* viscous effects (as stated from the outset), the only quantity of interest *is* τ_θ.

Proceeding in a similar manner, the case of high flow velocities, corresponding to a *turbulent boundary layer* may be treated. In this case

$$\tau_\theta = \tfrac{1}{2}\rho U^2 C_t (1/Re_a x)^{1/5}[-n\phi(\xi,1)/\bar{U}]\sin n\theta\, e^{\lambda\tau} \tag{26}$$

with $C_t = 0.0592$, as recommended by Schlichting.[16]

The Unsteady Flow in the Central Core

The unsteady flow in the axially nonuniform (because of boundary-layer growth) central core is governed by the same set of equations [Eqs. (7-12)] as in the case of inviscid flow, with the important difference that the boundary condition associated with Eq. (11) is now applied at $r=a-\delta^*(x)$ instead of at $r=a$. The displacement thickness $\delta^*(x)$ is given by Eq. (20).

Although the manipulations in this case are more complex, the same procedures are applied in Sec. II; for the sake of brevity, the final answer for $p-p_\infty$ will be given without further ado, i.e.,

$$\begin{aligned} p-p_\infty = -\rho U_{\rm ref}^2 \sum_m c_m \{ & \lambda^2[\eta_m^*(\xi)\phi_{mH}(\xi) + \zeta_m^*(\xi)\phi_{mT}(\xi)] \\ & + \bar{U}\lambda[2\eta_m^*(\xi)\phi_{mH}'(\xi) + 2\zeta_m^*(\xi)\phi_{mT}'(\xi) + \eta_m^{*\prime}(\xi)\phi_{mH}(\xi) \\ & + \zeta_m^{*\prime}(\xi)\phi_{mT}(\xi)] + \bar{U}^2[(\beta_m/\ell)^2\{\eta_m^*(\xi)\phi_{mH}(\xi) \\ & - \zeta_m^*(\xi)\phi_{mT}(\xi)\} + \eta_m^{*\prime}(\xi)\phi_{mH}'(\xi) + \zeta_m^{*\prime}(\xi)\phi_{mT}'(\xi)]\} \\ & \times \cos n\theta\, e^{\lambda\tau} \end{aligned} \tag{27}$$

where

$$\eta_m^*(\xi) = J_n\{(1-\bar{\delta}^*)\beta_m/\ell\}/\{(\beta_m/\ell)J_n'[(1-\bar{\delta}^*)\beta_m/\ell]\}$$

$$\zeta_m^*(\xi) = I_n\{(1-\bar{\delta}^*)\beta_m/\ell\}/\{(\beta_m/\ell)I_n'[(1-\bar{\delta}^*)\beta_m/\ell]\} \tag{28}$$

where the prime denotes the derivative with respect to the argument of the function, $\ell = L/a$, and $\bar{\delta}^* \doteq \delta^*/a$. The expression for $p-p_\infty$ above is more complex than that of Eq. (17), because the coefficients η_m^* and ζ_m^* are functions of ξ in this case.

IV. Stability Analysis and Results

The unsteady motion of the cylindrical shell under study is described by Eq. (3), where the effect of τ_x is neglected, and τ_θ and $p-p_\infty$ are given by Eqs. (25) or (26) and (27). After application of Galerkin's method, the solution is reduced to an eigenvalue problem, yielding the determinantal equation

$$\det[A_{qm}] = 0 \tag{29}$$

where the expression for A_{qm}, which is a sixth-order polynomial in λ_k, is not given here in the interests of brevity; Eq. (29) yields the eigenvalues of the problem, λ_k.

The matrix in Eq. (29) is nine times smaller than if the set of three equations of motion of the shell had been utilized—instead of the unique higher-order equation determined in Sec. II. It was found that this method was more quickly convergent and accurate than the three-equation method for the same number of comparison functions; hence, it was also computationally more efficient and 17 times faster, for comparable accuracy.[15]

Calculations were conducted for both inviscid and viscous flow, and the results were then compared. In the case of viscous flow the eigenvalues λ_k are no longer purely imaginary; they are complex, due to the unsteady viscous effects, although the real component (for flows smaller than the threshold of buckling) is quite small. Results for the second circumferential mode ($n=2$) for both inviscid and viscous flow are presented in Figs. 3 and 4, showing the variation of the dimensionless eigenfrequencies ω_k [$\omega_k = Im(\lambda_k)$] vs the dimensionless flow velocity $\bar{U}$, of two typical systems: the so-called air-rubber system and water-steel system, so named according to the material of the shell and the fluid being conveyed. The physical characteristics of the systems are given in Table 1. The calculations were conducted with four terms in the Galerkin expansion; it was confirmed that with this number of terms the results had converged with excellent accuracy and computational efficiency.

As may be seen in Figs. 3 and 4, the eigenfrequencies of all modes are diminished with the increasing $\bar{U}$; eventually, the first-mode eigenfrequencies vanish, at point B_1, indicating the onset of divergence (buckling) of the system. (In the case of viscous flow, strictly, point B_1 is slightly lower

Table 1 The physical characteristics of the systems of Figs. 3 and 4

System	ρ/ρ_s	$U_{\rm ref}$, m/s	ν	$\ell = L/a$	$\bar{h} = h/a$
Air-rubber	0.00136	36.73	0.5	25.9	0.0227
Water-steel	0.1282	5387	0.3	25.9	0.0227

Table 2 Critical flow velocities U_{B1} for inviscid and viscous flow

Analysis	Air-rubber	Water-steel
Inviscid	0.598	0.064
Viscous	0.612	0.065

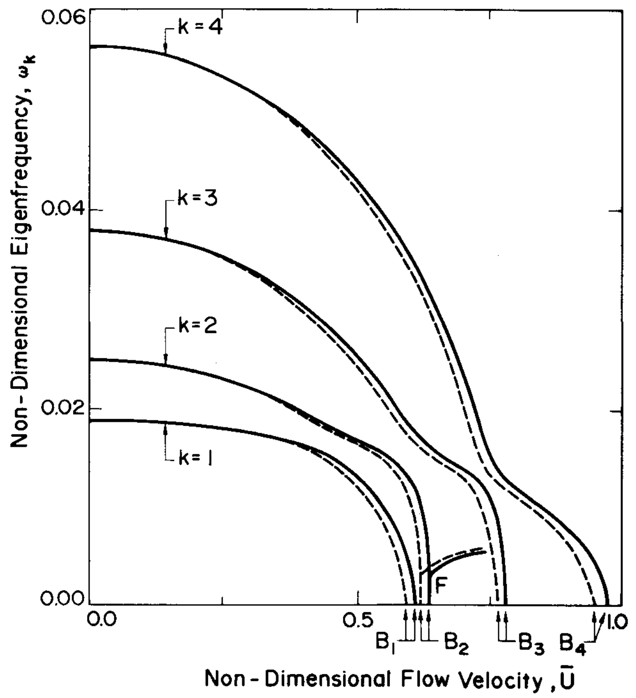

Fig. 3 The dimensionless eigenfrequencies, ω_k, of the first four axial modes of the system ($k=1,...,4$) and the second circumferential mode ($n=2$), as functions of $\bar{U}$ for the air-rubber system (Table 1). ———, theory taking into account unsteady viscous effects; - - -, inviscid theory.

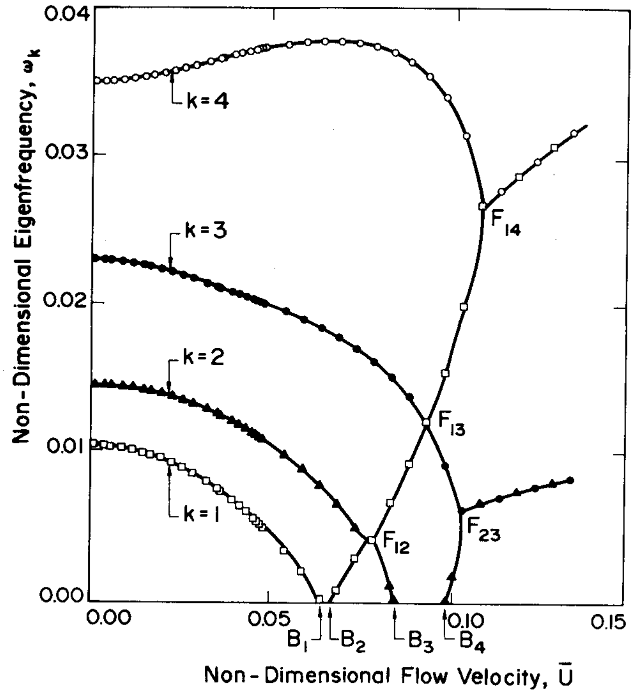

Fig. 4 The dimensionless eigenfrequencies, ω_k, for $n=2$ and $k=1,...,4$, as functions of $\bar{U}$ for the water-steel system (Table 1), according to the present theory, taking into account unsteady viscous effects.

than the buckling flow velocity, but the difference is negligibly small.) At that point, the first-mode eigenvalues are purely real, and $\omega_1=0$; beyond B_2, however, $\omega_1 \neq 0$ re-emerges.

Considering Fig. 3 in more detail, it is seen that for $\bar{U}$ slightly above point B_2, the first- and second-mode loci coalesce (at point F) giving rise to coupled-mode flutter. At higher $\bar{U}$, the system also becomes unstable by divergence in its third and fourth modes, at points B_3 and B_4. Of course, this post-critical (beyond B_1) behavior of the system does not necessarily materialize in practice, since the analysis is linear and the deformations associated with divergence are likely large. Nevertheless, flutter was observed in the experiments of Ref. 2.

In the case of Fig. 4, the inviscid and viscous results are very close, and only the latter are shown, for the sake of clarity. In this case, the post-critical behavior of the system is much more complex, involving coupling of the first- and second-mode loci at F_{12}, followed by a dissociation of this coupling almost immediately; this leads to divergence at point B_3. Other occurrences of coupled-mode flutter are associated with points F_{13}, F_{14}, and F_{23}.

Comparing inviscid and viscous results, it is seen that the eigenfrequencies ω_k are slightly larger in the viscous case. More importantly, the critical flow velocities in the viscous case are slightly higher, indicating that taking into account the presence of the boundary layer stabilizes the system (Table 2). This may be explained physically by the observation that the effective momentum of the flowing fluid is diminished by the presence of the boundary layer. It is noted that the differences between inviscid and viscous flow results are more pronounced for the higher modes, but in all cases remain small.

Comparing Fig. 3 with Fig. 4, it is seen that the detailed form of variation of ω_k with $\bar{U}$ is different; moreover the critical flow for divergence in the first mode, U_{B1}, is about 10 times lower for the water-steel system than for the air-rubber system (Table 2); however, the dimensional flow velocity in the latter case is actually lower, because of the ratio of $U_{\rm ref}$ in the two cases (Table 1).

Table 3 The effect of ℓ on stability

ℓ	5	10	15	25.9	35
$\bar{U}_{B1}$	1.26	0.791	0.684	0.598	0.582

Table 4 The effect of $\bar{h}$ on stability

$\bar{h}$	0.015	0.0227	0.035
$\bar{U}_{B1}$	0.415	0.598	0.895

The dynamic behavior and fluidelastic stability of the system are, of course, influenced by the whole set of parameters on which it depends. The influence of the nature of the fluid and the material of the shell has already been discussed. A further set of calculations has been undertaken to examine the effect of the geometric parameters $\ell = L/a$ and $\bar{h} = h/a$ on the air-rubber system. The results, insofar as the critical flow velocity for buckling, $\bar{U}_{B1}$, is concerned, are shown in Tables 3 and 4.

It is noted that if it is fairly small, the effect of increasing or decreasing ℓ is pronounced; for larger ℓ, however, the effect is rather weak (e.g., a 2.3-fold increase in ℓ, from 15 to 35, results in a reduction in $\bar{U}_{B1}$, by only 15%).

The effect of $\bar{h}$, examined in Table 4, in more pronounced, as expected.

V. Conclusions

The main objective of this paper was the investigation of the unsteady viscous effects, coupled with oscillations of the shell, on the dynamic behavior and stability of the shell—in the particular case of a developing boundary layer in the flow ingested by the shell.

Although full solution of the problem of a developing unsteady boundary layer would have been very difficult, it was possible to develop a technique, via which the unsteady problem was reduced to a quasisteady one. This was accomplished by making the reasonable assumptions that

1) shell oscillation amplitudes are small; 2) the mean flow velocity in the boundary layer remains sensibly constant in magnitude, but oscillates harmonically through a certain angle, synchronously with the circumferential movements of the shell surface. Although the modifications associated with taking into account the unsteady viscous effects appear to be simple at first sight, they nevertheless introduce important computational complications.

It was found that the unsteady viscous terms have a rather small effect on the dynamics and stability of the shell. Nevertheless, this is not to be viewed as a negative result, for it establishes, for the first time, that such effects are stabilizing and that they may be considered negligible from the practical viewpoint. (Here, it should be stressed once again that *steady* viscous effects, such as stresses associated with pressurization, to overcome the pressure drop, and surface traction were neglected in this paper—but have been shown to be generally important.[13])

An equally important contribution of this work has been associated with the formulation of the problem—thus transcending the main objective of this paper. The three equations of motion of the shell were reduced to a single equation of higher order, involving only the radial displacement of the shell. This was done without introducing any additional approximations to those associated with the original set of equations;[15] to the authors' knowledge, this also has been done for the first time. The advantage of this single-equation formulation was shown to be enhanced computational efficiency and considerable savings in cost. This formulation can be used in any problem involving circular cylindrical shells; for instance, it could be used as the basis of a finite element formulation in which the original three-dimensional problem is now reduced to a one-dimensional one.

Finally, the effect of a number of system parameters on stability has been investigated, in Sec IV. A more complete account of these calculations and all other aspects of the work described here may be found in Ref. 15.

Appendix: Final Equation of Motion

Starting with Flügge's set of three equations of motion,[14] and going through a rather complex process of successive differentiations and eliminations, the following single, eighth-order equation is obtained,[15] involving only the dimensionless displacement $\bar{w}$ and the externally imposed stress resultants:

$$\begin{aligned}
&\left\{A_1^1 \frac{\partial^4}{\partial\xi^4} + A_1^2 \frac{\partial^4}{\partial\xi^2\partial\theta^2} + A_1^3 \frac{\partial^4}{\partial\theta^4} + A_2^1 \frac{\partial^6}{\partial\xi^6}\right.\\
&+ A_2^2 \frac{\partial^6}{\partial\xi^4\partial\theta^2} + A_2^3 \frac{\partial^6}{\partial\xi^2\partial\theta^4} + A_2^4 \frac{\partial^6}{\partial\theta^6} + A_3^1 \frac{\partial^8}{\partial\xi^8}\\
&\left.+ A_3^2 \frac{\partial^8}{\partial\xi^6\partial\theta^2} + A_3^3 \frac{\partial^8}{\partial\xi^4\partial\theta^4} + A_3^4 \frac{\partial^8}{\partial\xi^2\partial\theta^6} + A_3^5 \frac{\partial^8}{\partial\theta^8}\right\}\bar{w}\\
&- \left\{A_4^1 \frac{\partial^4}{\partial\xi^4} + A_4^2 \frac{\partial^4}{\partial\xi^2\partial\theta^2} + A_4^3 \frac{\partial^4}{\partial\theta^4}\right\}(p-p_e)\\
&+ \left\{-A_4^4 \frac{\partial^3}{\partial\xi^3} + A_4^5 \frac{\partial^3}{\partial\xi\partial\theta^2} + A_4^6 \frac{\partial^5}{\partial\xi^5} - A_4^7 \frac{\partial^5}{\partial\xi^3\partial\theta^2}\right.\\
&\left.- A_4^8 \frac{\partial^5}{\partial\xi\partial\theta^4}\right\}\tau_x - \left\{A_4^9 \frac{\partial^3}{\partial\xi^2\partial\theta} + A_4^{10} \frac{\partial^3}{\partial\theta^3}\right.\\
&\left.- A_4^{11} \frac{\partial^5}{\partial\xi^4\partial\theta} - A_4^{12} \frac{\partial^5}{\partial\xi^2\partial\theta^3}\right\}\tau_\theta + \left\{A_5^1 \frac{\partial^6}{\partial\tau^6}\right\}\bar{w}\\
&+ \frac{\partial^4}{\partial\tau^4}\left\{A_6^1 - A_6^2 \frac{\partial^2}{\partial\xi^2} + A_6^3 \frac{\partial^2}{\partial\theta^2} + A_6^4 \nabla^4\right\}\bar{w}\\
&+ \frac{\partial^2}{\partial\tau^2}\left\{A_7^1 \frac{\partial^2}{\partial\xi^2} + A_7^2 \frac{\partial^2}{\partial\theta^2} + A_7^3 \frac{\partial^4}{\partial\xi^4} + A_7^4 \frac{\partial^4}{\partial\xi^2\partial\theta^2}\right.\\
&+ A_7^5 \frac{\partial^4}{\partial\theta^4} + A_7^6 \frac{\partial^6}{\partial\xi^6} + A_7^7 \frac{\partial^6}{\partial\xi^4\partial\theta^2} + A_7^8 \frac{\partial^6}{\partial\xi^2\partial\theta^4}\\
&\left.- A_7^9 \frac{\partial^6}{\partial\theta^6}\right\}\bar{w} - \left\{A_8^1 \frac{\partial^4}{\partial\tau^4}\right\}(p-p_e) + \frac{\partial^2}{\partial\tau^2}\left\{A_9^1 \frac{\partial^2}{\partial\xi^2}\right.\\
&\left.+ A_9^2 \frac{\partial^2}{\partial\theta^2}\right\}(p-p_e) + \frac{\partial^3}{\partial\xi\partial\tau^2}\left\{A_9^3 - A_9^4 \frac{\partial^2}{\partial\xi^2}\right.\\
&\left.+ A_9^5 \frac{\partial^2}{\partial\theta^2}\right\}\tau_x + \frac{\partial^3}{\partial\theta\partial\tau^2}\left\{A_9^6 - A_9^7 \frac{\partial^2}{\partial\xi^2}\right\}\tau_\theta = 0
\end{aligned}$$

The coefficient denoted by A_i^j, $i=1,2,\ldots,9$ and $j=1,2,\ldots,12$ are given by

$$A_1^1 = (1+3k)(1+k-\nu^2)$$

$$A_1^2 = k[2+(1-\nu)(1+k)(4+3k)/2]$$

$$A_1^3 = k(1+k)$$

$$A_2^1 = 2k\nu(1+3k), \quad A_2^2 = 3k[2+(\nu^2-\nu+2)k]$$

$$A_2^3 = k\{8-2\nu+k[2+(1-\nu)(5+3k)]\}$$

$$A_2^4 = 2k(1+k)$$

$$A_3^1 = k(1-k)(1+3k)$$

$$A_3^2 = k[4(1+k)+3k(1-\nu)(1+3k)/2]$$

$$A_3^3 = k[6(1+k)-k\nu(3+k\nu)]$$

$$A_3^4 = k\{2+(1+k)[2+3k(1-\nu)/2]\}, \quad A_3^5 = k(1+k)$$

$$A_4^1 = (1-\nu^2)(1+3k)/E\bar{h}$$

$$A_4^2 = (1-\nu^2)[2+k(1-\nu)(4+3k)/2]/E\bar{h}$$

$$A_4^3 = (1-\nu^2)(1+k)/E\bar{h}$$

$$A_4^4 = \nu A_4^1, \quad A_4^5 = (1-\nu^2)/E\bar{h}, \quad A_4^6 = kA_4^1$$

$$A_4^7 = (1-\nu^2)3k^2(1-\nu)/2E\bar{h}, \quad A_4^8 = kA_4^5$$

$$A_4^9 = (1-\nu^2)(2+\nu)/E\bar{h}, \quad A_4^{10} = A_4^3, \quad A_4^{11} = 2kA_4^5$$

$$A_4^{12} = (1-\nu^2)k[2+k(3-\nu)/2]/E\bar{h}$$

$$A_5^1 = 2/(1-\nu)$$

$$A_6^1 = 2(1+k)/(1-\nu), \quad A_6^2 = [3-\nu+3k(1-\nu)]/(1-\nu)$$

$$A_6^3 = [\nu(1+k)-3(1-k)]/(1-\nu), \quad A_6^4 = 2k(1-\nu)$$

$$A_7^1 = \{2\nu^2-(1+k)[3-\nu+3k(1-\nu)]\}/(1-\nu)$$

$$A_7^2 = \{2-(1+k)[3-\nu+k(1-\nu)]\}/(1-\nu)$$

$$A_7^3 = 1+3k-4k\nu/(1-\nu)$$

$$A_7^4 = 2-k[2(5-\nu)/(1-\nu)+3k(3+\nu)/2]$$

$$A_7^5 = 1-k[2(1+k)+(3+\nu)/(1-\nu)]$$

$$A_7^6 = k[(3\nu-1)k-(3-\nu)]/(1-\nu)$$

$$A_7^7 = k[k(\nu^2+12\nu-9)/2-3(3-\nu)]/(1-\nu)$$

$$A_7^8 = k[k(\nu^2 + 8\nu - 9)/2 - 3(3 - \nu)]/(1 - \nu)$$

$$A_7^9 = k[(3 - \nu)/(1 - \nu) + k]$$

$$A_8^1 = (1 + \nu)a/2E\bar{h}$$

$$A_9^1 = (1 + \nu)a[3 - \nu + 3k(1 - \nu)]/4E\bar{h}$$

$$A_9^2 = (1 + \nu)a[3 - \nu + k(1 - \nu)]/4E\bar{h}, \quad A_9^3 = \nu A_9^6$$

$$A_9^4 = kA_9^6, \; A_9^5 = (1 - \nu)A_9^4/2, \; A_9^6 = (1 + \nu)a/2E\bar{h}$$

$$A_9^7 = (3 - \nu)A_9^4/2$$

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